# Elliptic Curves

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- Their solution sets have a special property.

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- The square roots are real numbers iff

$$x^3 + ax + b \geq 0.$$

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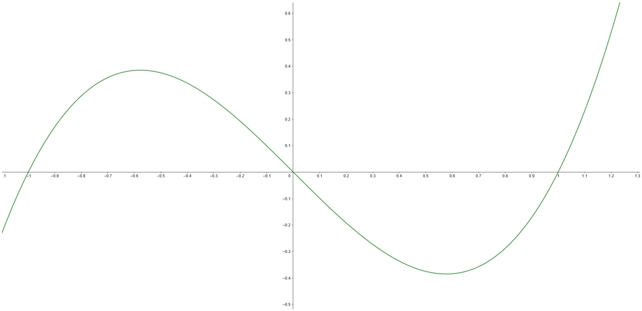
- If $\Delta > 0$, then the cubic has three distinct real roots:



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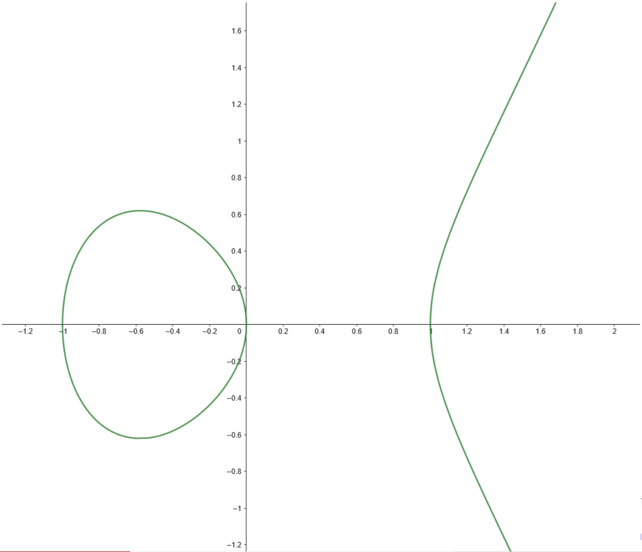

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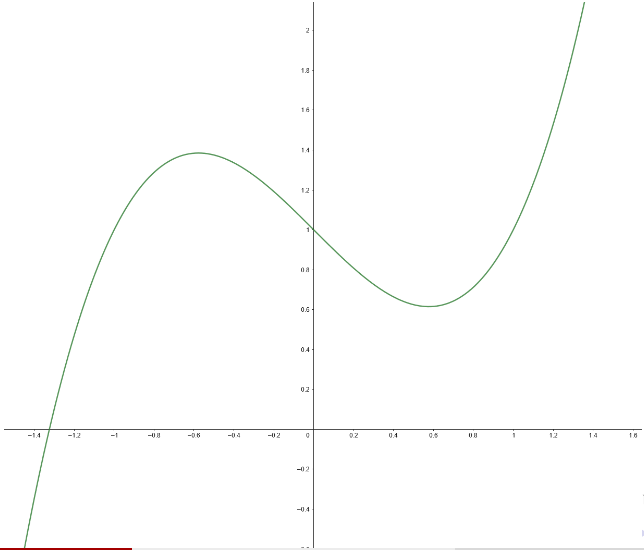

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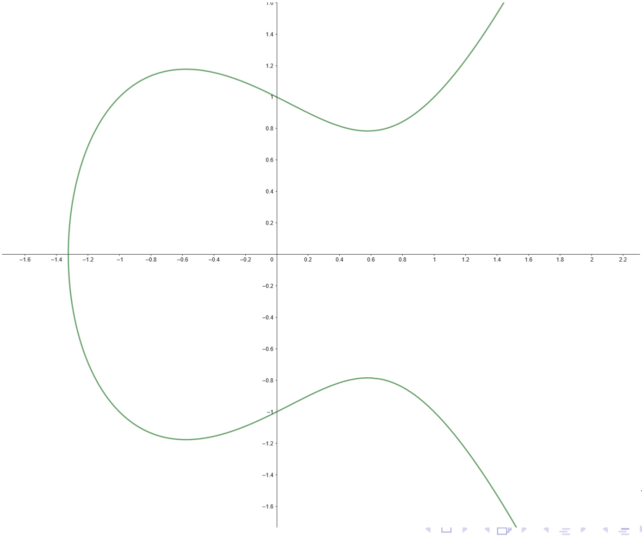

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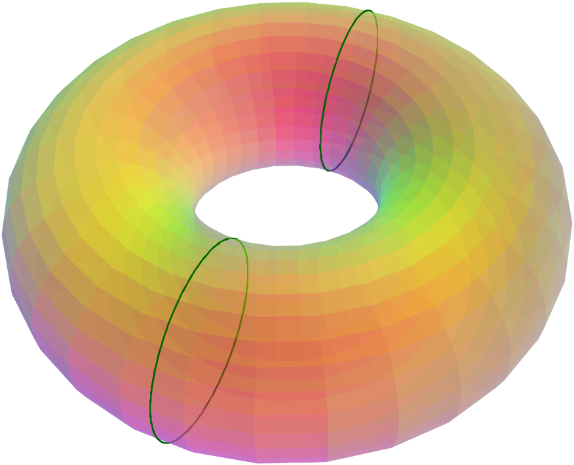
- The circles are a slice of the solutions over $\mathbb{C}$.



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- Much more difficult! Still an active research area.
- Can you find solutions to

$$y^2 = x^3 - x + 1$$

that are integers or rational numbers?

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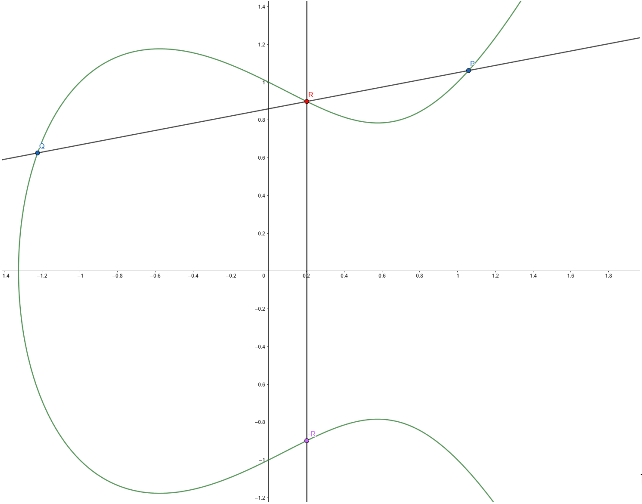
- Things to ponder:
  - What if $Q = P$?
  - What if $Q = -P$?
  - What is the identity element?

- Example:



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  - Polynomial long division or Vieta's formulas may help.

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- We'll handle the case $P = Q$ later.

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- The coefficient of $x^2$ is

$$-m^2 = -(x_1 + x_2 + x_3) \implies x_3 = m^2 - (x_1 + x_2).$$

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- Plug $x_3$ back into the equation of the line:

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- The final formula is $P + Q = (x_3, y_3)$, where

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$x_3 = m^2 - (x_1 + x_2)$$
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- This formula doesn't work at $y_1 = 0$.
- Why not? If $P = (x_1, 0)$, then what is $2P$?

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- Other curves may have more complicated groups of rational points (or no rational points at all!).
- Easier exercise: find all solutions to

$$y^2 \equiv x^3 - x + 1 \pmod 5.$$

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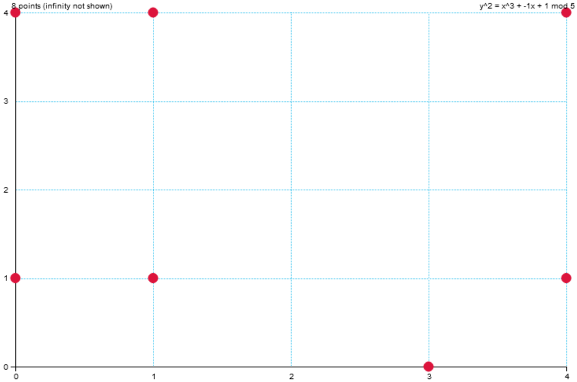
# Solutions over $\mathbb{Q}$ and Finite Fields

- Our curve looks like this over $\mathbb{F}_5$:



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- Elliptic curves were a crucial ingredient in Andrew Wiles' proof of Fermat's Last Theorem (FLT).
- FLT says there are no integer solutions to the equation

$$a^n + b^n = c^n$$

where $n > 2$ and $a, b, c$ are all nonzero.

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